Laguerre Entire Functions and Related Locally Convex Spaces

by

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Abstract

A scale $\{\mathcal{A}_a,\ a \geq 0\}$, $\mathcal{A}_a \subset \mathcal{A}_b$, for $a \leq b$, of the Fréchet spaces of exponential type entire functions of one complex variable is considered. Certain special properties of the subsets of $\mathcal{A}_a$ consisting of Laguerre entire functions, which are obtained as uniform limits on compact subsets of $\mathbb{C}$ of polynomials with real nonpositive zeros only, are described. On the space $\mathcal{A}_b$, the operators having the form $\varphi(\Delta_\theta)$, where $\varphi \in \mathcal{A}_a$, $ab < 1$, and $\Delta_\theta = (\theta + zD)D$ with $\theta \geq 0$ and $D = d/dz$, are defined. They are shown to preserve the set of Laguerre entire functions $\mathcal{L}^+$. An integral form of $\exp(a\Delta_\theta)$ with $a > 0$ is found that allows to construct some extensions of this operator. These results are used to obtain and to study the solutions of a certain initial value problem involving Δ_θ.

1 Introduction and Main Results

1.1 Introduction

In this paper, topological vector spaces of exponential type entire functions of one complex variable are considered. We introduce a scale of spaces $\{\mathcal{A}_a,\ a \geq 0\}$, $\mathcal{A}_a \subset \mathcal{A}_b$, for $a \leq b$, where each $\mathcal{A}_a$ is defined as a Fréchet space. Applying some results of [12] we show that the set of all polynomials is dense in every $\mathcal{A}_a$ and that the relative topology on every bounded subset $B \subset \mathcal{A}_a$ coincides with the topology of uniform convergence on compact subsets of $\mathbb{C}$ (Theorem 1.1). This implies that each $\mathcal{A}_a$ inherits Montel's property from the space of all entire functions, i.e., its bounded subsets are precisely the

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relatively compact subsets. It also means if a sequence $\{f_n\} \subset \mathcal{A}_a$ (a) converges uniformly on compact subsets of $\mathbb{C}$ to a function f; (b) this f belongs to $\mathcal{A}_a$; (c) it is bounded in $\mathcal{A}_a$, then this sequence converges to f also in $\mathcal{A}_a$. We study the possibility to slack the above conditions by excluding (c) and to obtain it as a result of the convergence $f_n \to f$ in $\mathcal{A}_a$. It turns out that this generally is impossible and we give some examples of sequences $\{f_n\} \in \mathcal{A}_a$ converging to $f \in \mathcal{A}_a$ uniformly on compact subsets of $\mathbb{C}$, which are unbounded in $\mathcal{A}_a$. At the same time we prove that this possibility occurs (Theorem 1.2) for the sequences $\{f_n\} \subset \mathcal{L}^+$. The latter is a class of Laguerre entire functions. These functions are obtained as uniform limits on compact subsets of $\mathbb{C}$ of the sequences of polynomials possessing real nonpositive zeros only. The class $\mathcal{L}^+$ was being studied by many authors during all this century, that was caused by a number of significant properties and various applications of these functions. A considerable survey of this study is given in [6] (see also [8]).

On the introduced Fréchet spaces $\mathcal{A}_b$, we define the operators having the form $\varphi(\Delta_\theta)$, where $\varphi \in \mathcal{A}_a$, $ab < 1$, and $\Delta_\theta = (\theta + zD)D$ with $\theta \geq 0$ and $D = d/dz$. It is proven that each such an operator continuously maps $\mathcal{A}_b \to \mathcal{A}_c$ with $c = b(1-ab)^{-1}$. The main result obtained here is Theorem 1.3 which asserts that if $\varphi \in \mathcal{L}^+$ and $f \in \mathcal{L}^+$, then $\varphi(\Delta_\theta)f \in \mathcal{L}^+$ provided it exists in some $\mathcal{A}_c$. To prove this assertion we construct a technique in the form of Lemma 2.1 – Lemma 2.4, which allows us to control the distribution of zeros of the function $\varphi(\Delta_\theta)f$. Further we find an integral form of $\exp(a\Delta_\theta)$, $a \geq 0$ (Proposition 1.5), which together with an analog of the operation rules (Proposition 1.6) allows to construct some extensions of this operator. This is then used to obtain and to study the solutions of a certain initial value problem involving Δ_θ (Theorem 1.5).

All simple proofs follow directly the statements. More complicated and more technical proofs are placed in the second section.

1.2 Spaces of Entire Functions

Let $\mathcal{E}$ be the set of all entire functions $\mathbb{C} \to \mathbb{C}$ equipped with the pointwise linear operations and with the topology $\mathcal{T}_C$ of uniform convergence on compact subsets of $\mathbb{C}$. For $b > 0$, we define

$$\mathcal{B}_b = \{f \in \mathcal{E} \mid \|f\|_b < \infty\};$$

where

$$\|f\|_b = \sup_{k\in\mathbb{N}_0}\{b^{-k} \mid f^{(k)}(0) \mid\}; \quad f^{(k)}(0) = (D^k f)(0) = \frac{d^k f}{dz^k}(0), \tag{1.1}$$

and $\mathbb{N}_0$ stands for the set of all nonnegative integers.

Proposition 1.1 *Every $(\mathcal{B}_b, \|\cdot\|_b)$ is a Banach space. Every sequence $\{f_n\}$ converging in $\mathcal{B}_b$, converges also in $(\mathcal{E}, \mathcal{T}_C)$.*

Proof. To prove the first part of this statement we need only to show the completeness of $\mathcal{B}_b$. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{B}_b$, K be a compact subset of $\mathbb{C}$, and

$$r(K) \stackrel{\text{def}}{=} \sup_{z\in K}|z|.$$

Given $\varepsilon > 0$, we choose N such that for all $m, p > N$,

$$\|f_m - f_p\|_b < \varepsilon, \tag{1.2}$$

and find

$$\begin{aligned} \sup_{z\in K}|f_m(z) - f_p(z)| &= \sup_{z\in K}\left|\sum_{k=0}^{\infty}\frac{f_m^{(k)}(0) - f_p^{(k)}(0)}{k! b^k}(bz)^k\right| \\ &\leq \|f_m - f_p\|_b \sup_{z\in K}\sum_{k=0}^{\infty}\frac{|bz|^k}{k!} \leq \varepsilon \exp(br(K)), \end{aligned}$$

which means that $\{f_n\}$ is a Cauchy sequence in $(\mathcal{E}, \mathcal{T}_C)$. Hence there exists an entire function f such that $f_n \to f$ in $(\mathcal{E}, \mathcal{T}_C)$. This proves the second part of the statement. The convergence just established means that, for every $k \in \mathbb{N}$, the sequence $\{f_n^{(k)}(0)\}$ converges to $f^{(k)}(0)$ (by Weierstrass' theorem). On the other hand, for every $k \in \mathbb{N}_0$, one easily gets from (1.1), (1.2)

$$b^{-k}\left|f_m^{(k)}(0) - f_p^{(k)}(0)\right| \leq \|f_m - f_p\|_b < \varepsilon. \tag{1.3}$$

Thus passing here to the limit $p \to \infty$ one obtains

$$\sup_{k\in\mathbb{N}_0}\left\{b^{-k}\left|f_m^{(k)}(0) - f^{(k)}(0)\right|\right\} \leq \varepsilon. \tag{1.4}$$

Hence

$$\begin{aligned}\sup_{k\in\mathbb{N}_0}\left\{b^{-k}\left|f^{(k)}(0)\right|\right\} &\le \sup_{k\in\mathbb{N}_0}\left\{b^{-k}\left|f_m^{(k)}(0)\right|\right\}+\sup_{k\in\mathbb{N}_0}\left\{b^{-k}\left|f_m^{(k)}(0)-f^{(k)}(0)\right|\right\}\\ &\le \|f_m\|_b+\varepsilon,\end{aligned}$$

which means $f\in\mathcal{B}_b$. Then the estimate (1.4) implies

$$\|f_m-f\|_b\le\varepsilon<2\varepsilon,$$

thus $f_m\to f$ in $\mathcal{B}_b$. ■

The central role in this work is played by the following space of entire functions. For $a\ge 0$, let

$$\mathcal{A}_a=\bigcap_{b>a}\mathcal{B}_b=\{f\in\mathcal{E}\mid(\forall b>a)\ \|f\|_b<\infty\}, \tag{1.5}$$

This set equipped with the topology $\mathcal{T}_a$ defined by the family $\{\|.\|_b,\ b>a\}$ becomes a separable locally convex space. Due to the following obvious inequality

$$\|f\|_{b'}<\|f\|_b,\qquad b'>b$$

the topology $\mathcal{T}_a$ may also be defined by a countable family of norms $\{\|.\|_{b_n},n\in\mathbb{N}\}$ (e.g. $b_n=a+1/n$). Thus the topological vector space $(\mathcal{A}_a,\mathcal{T}_a)$, being the projective limit of a countable family of the Banach spaces $(\mathcal{B}_{b_n},\|.\|_{b_n})$, is complete and metrizable (see [11], chap.I, § 6 and chap.II, § 5). Therefore $(\mathcal{A}_a,\mathcal{T}_a)$ is a Fréchet space. In the sequel we will write $\mathcal{E}$, $\mathcal{A}_a$, and $\mathcal{B}_b$ instead of $(\mathcal{E},\mathcal{T}_C)$, $(\mathcal{A}_a,\mathcal{T}_a)$, and $(\mathcal{B}_b,\|.\|_b)$ respectively, assuming that the mentioned topologies are the standard ones on these sets. It should be pointed out that $\mathcal{E}$ and $\mathcal{A}_0$, equipped also with the pointwise multiplication, become algebras. As a subset of $\mathcal{E}$, $\mathcal{A}_a$ equipped with $\mathcal{T}_a$ should inherit some properties of $(\mathcal{E},\mathcal{T}_C)$. An important example here is Montel's property – the bounded subsets of $\mathcal{E}$ are precisely the relatively compact subsets (see [1] p. 120). A subset $B\subset\mathcal{E}$ is said to be bounded if for every compact $K\in\mathbb{C}$, there exists $C(K)$ such that

$$\sup_{f\in B}\{\sup_{z\in K}|f(z)|\}\le C(K).$$

Definition 1.1 *A subset $B \subset \mathcal{A}_a$ is said to be bounded in $\mathcal{A}_a$ if for every $b > a$, there exists a constant C_b such that*

$$\sup_{f \in B} \|f\|_b \leq C_b.$$

Theorem 1.1 *For every $a \geq 0$, the space $\mathcal{A}_a$ possesses the properties:*

(i) *the set of all polynomials $\mathcal{P} \subset \mathcal{E}$ is dense in $\mathcal{A}_a$;*

(ii) *the relative topology on every bounded subset $B \subset \mathcal{A}_a$ coincides with the topology of uniform convergence on compact subsets of $\mathbb{C}$;*

(iii) *for every $f \in \mathcal{A}_a$ and $g \in \mathcal{A}_b$, their product fg belongs to $\mathcal{A}_{a+b}$.*

Corollary 1.1 *A subset $B \in \mathcal{A}_a$, which is closed and bounded in $\mathcal{E}$, is closed in $\mathcal{A}_a$ provided it is bounded there.*

Corollary 1.2 [Montel's Property] *The bounded subsets of $\mathcal{A}_a$ are exactly the relatively compact subsets.*

Corollary 1.3 *Let $\{f_n,\ n \in I\!N\}$ converge in $\mathcal{A}_a$ to f and $\{g_n,\ n \in I\!N\}$ converge in $\mathcal{A}_b$ to g. Then the sequence $\{f_n g_n,\ n \in I\!N\}$ converges to fg in $\mathcal{A}_{a+b}$.*

It may seem that a sequence $\{f_n, n \in I\!N\} \subset \mathcal{A}_a$ with the following properties: (a) it converges in $\mathcal{E}$ to a function f; (b) $f \in \mathcal{A}_a$; should possess the property: (c) it converges to this f also in $\mathcal{A}_a$. In view of claim (ii) of the above theorem, to prove this conjecture it would be enough to show the boundedness of this sequence in $\mathcal{A}_a$. In fact, the conjecture is false. Furthermore, a claim like (ii) does not hold for the Banach spaces $\mathcal{B}_b$. The following examples show the subtlety of the situation with the topologies of the mentioned spaces. Consider the sequence $\{f_n(z) = z^n/n!,\ n \in I\!N\}$. Since it consists of polynomials, it is a subset of $\mathcal{A}_0$ and of all $\mathcal{B}_b$, $b > 0$. By means of (1.1), one finds $\|f_n\|_a = a^{-n}$. Thus the sequence is bounded only in $\mathcal{A}_a$ and in $\mathcal{B}_a$ with $a \geq 1$. On the other hand, this sequence converges in $\mathcal{E}$ to the function $f(z) \equiv 0$ which also belongs to all mentioned spaces. In all

$\mathcal{A}_a$, $a \geq 1$, where it is bounded, the sequence converges to $f \equiv 0$, as it is prescribed by claim (ii). In $\mathcal{A}_a$, $a < 1$ our sequence possesses (a) and (b) but is unbounded in $\mathcal{A}_a$ hence does not possess (c). Moreover, it is bounded in $\mathcal{B}_1$ and does not converge in this space, i.e., a claim like (ii) would fail for the Banach spaces $\mathcal{B}_b$. Another example, which is only a slight variation of the previous one, is as follows. Let f be an arbitrary element of $\mathcal{A}_a$, $a < 1$. For $\varepsilon > 0$, we define the sequence of functions $\{f_n(z, \varepsilon)\}$ by their derivatives at $z = 0$:

$$f_n^{(k)}(0, \varepsilon) = f^{(k)}(0) + \varepsilon \delta_{kn}, \quad k, n \in \mathbb{N}_0,$$

where δ stands for the Kronecker δ–symbol. Then this sequence converges to f in every $\mathcal{A}_a$, $a \geq 1$, and hence does in $\mathcal{E}$. But, for every positive ε, it is unbounded in any $\mathcal{A}_a$ with $a < 1$.

Nevertheless, there exists the subset of $\mathcal{E}$ such that on its intersections with the spaces $\mathcal{A}_a$, $a \geq 0$, the above mentioned properties (a) and (b) imply (c).

Definition 1.2 *A family $\mathcal{L}$ (respectively $\mathcal{L}_0$, $\mathcal{L}^+$, $\mathcal{L}^-$) consists of all entire functions possessing the following representation*

$$f(z) = Cz^l \exp(\alpha z) \prod_{j=1}^{\infty} (1 + \beta_j z); \tag{1.6}$$

$$C \in \mathbb{C}; \; l \in \mathbb{N}_0; \quad \beta_j \geq \beta_{j+1} \geq 0; \; \sum_{j=1}^{\infty} \beta_j < \infty,$$

with $\alpha \in \mathbb{R}$ (respectively $\alpha = 0$, $\alpha \geq 0$, and $\alpha < 0$).

The functions which form $\mathcal{L}^+$ are known as the Laguerre entire functions [6]. Due to Laguerre and Pólya (see e.g. [6], [8]), we know that

Proposition 1.2 *The family $\mathcal{L}^+$ consists of all entire functions which are the polynomials possessing real nonpositive zeros only or their uniform limits on compact subsets of $\mathbb{C}$.*

We denote by $\mathcal{P}^+$ the set of polynomials belonging to $\mathcal{L}^+$. It should be pointed out that a function f, possessing the representation (1.6) with given $\alpha \in \mathbb{R}$, belongs to $\mathcal{B}_b$ with $b > |\alpha|$, thus it belongs to $\mathcal{A}_{|\alpha|}$ (but it may not belong to $\mathcal{B}_{|\alpha|}$). It is also worth to remark that every f being of the form

(1.6) may be written $f(z) = \exp(\alpha z)h(z)$, where h is an entire function of exponential type zero (for the details see e.g. [2], [8]).

Consider the families

$$\mathcal{L}_a \stackrel{\text{def}}{=} \mathcal{L} \cap \mathcal{A}_a, \quad \mathcal{L}_a^{\pm} \stackrel{\text{def}}{=} \mathcal{L}^{\pm} \cap \mathcal{A}_a. \tag{1.7}$$

Obviously the latter definition of $\mathcal{L}_0$ coincides with that given by Definition 1.2, and $\mathcal{L}_0 = \mathcal{L}_0^+$.

Now let us return to the example considered just after Theorem 1.1. The sequence $\{z^n/n!\}$, as well as its limit in $\mathcal{E}$, $f \equiv 0$, belong to $\mathcal{L}_0$, thus the implication (a) and (b) $\Rightarrow$ (c) fails also on $\mathcal{L}_0$. But this is the unique example of $f \in \mathcal{L}^+$ of this type.

Theorem 1.2 *Every sequence of functions $\{f_n, n \in I\!N\} \subset \mathcal{L}_a^+$, $a \geq 0$, that converges in $\mathcal{E}$ to a function $f \in \mathcal{A}_a$, which does not vanish identically, is a bounded subset of this $\mathcal{A}_a$ and hence converges to f also in $\mathcal{A}_a$. The limit function belongs to $\mathcal{L}^+$ as well.*

Corollary 1.4 *A subset $B \subset \mathcal{L}_a^+$, which is compact in $\mathcal{E}$, is also compact in $\mathcal{A}_a$ provided it does not contain $f \equiv 0$.*

Corollary 1.5 *For every $a \geq 0$, the set $\mathcal{P}^+$ is dense in $\mathcal{L}_a^+$ in $\mathcal{A}_a$.*

Now we establish some sufficient conditions for a sequence in $\mathcal{L}$ to be bounded in some $\mathcal{A}$ not assuming its convergence. Let $\{f_n, \ n \in I\!N\}$ be a sequence of functions from $\mathcal{L}$. Denote by C_n, l_n, α_n, and $\beta_j(n)$ the corresponding parameters of f_n in its representation (1.6). Let also

$$\mu_k(n) \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \beta_j^k(n), \quad k \in I\!N. \tag{1.8}$$

Proposition 1.3 *Given a sequence $\{f_n, \ n \in I\!N\} \in \mathcal{L}$, let there exist positive a, C, and $l \in I\!N_0$ such that, for all $n \in I\!N$,*

$$|\alpha_n| + \mu_1(n) \leq a; \quad |C_n| \leq C; \quad l_n \leq l.$$

Then this sequence $\{f_n, n \in I\!N\}$ is bounded in $\mathcal{A}_a$.

Proof. For $k < l_n$, $f_n^{(k)}(0) = 0$. For $k \geq l_n$, a simple calculation based on the representation (1.6) yields

$$\begin{aligned} f_n^{(k)}(0) &= C_n \frac{k!}{(k-l_n)!} \sum_{i=0}^{k-l_n} \left[\binom{k-l_n}{i} \alpha_n^{k-l_n-i} \right. \\ &\qquad \left. i! \sum_{1 \leq j_1 < j_2 < \ldots < j_i < \infty} \beta_{j_1}(n) \beta_{j_2}(n) \ldots \beta_{j_i}(n) \right]. \end{aligned}$$

Then

$$\begin{aligned} \left| f_n^{(k)}(0) \right| &\leq C k^{l_n} \sum_{i=0}^{k-l_n} \binom{k-l_n}{i} |\alpha_n|^{k-l_n-i} \left(\mu_1(n)\right)^i \\ &= C k^{l_n} \left(|\alpha_n| + \mu_1(n)\right)^{k-l_n} \leq C (k/a)^{l_n} a^k. \end{aligned}$$

Thus for $b > a$,

$$\|f_n\|_b = \sup_{k \in I\!N_0} \left\{ b^{-k} \left| f_n^{(k)}(0) \right| \right\} \leq \sup_{k \in I\!N_0} \left\{ C (k/a)^{l_n} (a/b)^k \right\}. \tag{1.9}$$

Finally, in view of the fact $l_n \leq l$, the above estimate implies that there exists a constant K depending only on a and b such that $\|f_n\|_b \leq K$. ∎

1.3 Operators

For $\theta \geq 0$, let us consider a map $\Delta_\theta : \mathcal{E} \to \mathcal{E}$ defined as follows

$$(\Delta_\theta f)(z) = (\theta + zD)Df(z) = \theta \frac{df}{dz} + z \frac{d^2 f}{dz^2}. \tag{1.10}$$

The study of this map is quite important in view of the following facts. First, one observes that, for $g(z) = f(z^2)$,

$$(\Delta_\theta f)(z^2) = \frac{1}{4} \left(\frac{2\theta - 1}{z} \frac{dg(z)}{dz} + \frac{d^2 g(z)}{dz^2} \right), \tag{1.11}$$

which means that, for $\theta = N/2$, $N \in I\!N$, the map (1.10) is connected by the latter identity with the radial part Δ_r of the N–dimensional Laplacean

$$\Delta_r = \frac{N-1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}.$$

This connection will be used by us in a separate work. Another application arises from the fact that this map may produce the Laguerre polynomials, which usually are defined as follows (see e.g. [10], p. 147)

$$\tilde{L}_n^{(\theta-1)}(z) \stackrel{\text{def}}{=} (-1)^n z^{-\theta+1} e^z \left(D^n z^{\theta+n-1} e^{-z} \right), \tag{1.12}$$

namely

$$\tilde{L}_n^{(\theta-1)}(z) = e^z \Delta_\theta^n e^{-z} = \exp(-\Delta_\theta) z^n. \tag{1.13}$$

The latter formula contains an expression which in general situations needs to be defined more precisely.

For given two entire functions φ and f, we denote by $\varphi(\Delta_\theta)f(z)$ the formal series

$$\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{\varphi^{(k)}(0)}{k!} \frac{f^{(m)}(0)}{m!} \Delta_\theta^k z^m. \tag{1.14}$$

One can verify that

$$\Delta_\theta^k z^m = q_\theta^{(m,k)} z^{m-k}, \qquad q_\theta^{(m,k)} = \begin{cases} 0, & k > m \\ \gamma_\theta(m)/\gamma_\theta(m-k), & 0 \le k \le m \end{cases},$$

where

$$\gamma_\theta(m) = m!\Gamma(\theta+m). \tag{1.15}$$

Proposition 1.4 *For positive a and b obeying the condition $ab < 1$, let $\varphi \in \mathcal{B}_a$ and $f \in \mathcal{B}_b$. Then, for every $\theta \ge 0$, the function $g(z) = \varphi(\Delta_\theta)f(z)$ belongs to $\mathcal{B}_c$ with $c = b(1-ab)^{-1}$. Furthermore*

$$\|g\|_c \le (1-ab)^{-\theta} \|\varphi\|_a \|f\|_b. \tag{1.16}$$

Proof According to (1.14)

$$g(z) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} z^n,$$

with

$$g^{(n)}(0) = \sum_{k=0}^{\infty} \frac{n!}{k!(n+k)!} \varphi^{(k)}(0) f^{(n+k)}(0) q_\theta^{(n+k,k)}. \tag{1.17}$$

But $\left|\varphi^{(k)}(0)\right| \leq a^k \|\varphi\|_a$ and $\left|f^{(m)}(0)\right| \leq b^m \|f\|_b$ (see (1.1)). For positive a and b obeying the condition $ab < 1$, one may show that

$$\sum_{k=0}^{\infty} \frac{n!}{k!(n+k)!}(ab)^k q_\theta^{(n+k,k)} = (1-ab)^{-n-\theta},$$

which yields in (1.17)

$$\begin{aligned} \left|g^{(n)}(0)\right| &\leq \sum_{k=0}^{\infty} \frac{n!}{k!(n+k)!} \left|\varphi^{(k)}(0)\right| \left|f^{(n+k)}(0)\right| q_\theta^{(n+k,k)} \\ &\leq (1-ab)^{-\theta} \|\varphi\|_a \|f\|_b \left(\frac{b}{1-ab}\right)^n . \end{aligned}$$

Hence $g \in \mathcal{B}_c$ and the estimate (1.16) holds. ∎

Corollary 1.6 *For all $\theta \geq 0$, $a \geq 0$, and $b \geq 0$, such that $ab < 1$, $(\varphi, f) \mapsto \varphi(\Delta_\theta)f$ is a continuous bilinear map from $\mathcal{A}_a \times \mathcal{A}_b$ into $\mathcal{A}_c$, where $c = b(1-ab)^{-1}$.*

The following strengthening of the above statement is one of the main results of this research.

Theorem 1.3 *For all $\theta \geq 0$, $a \geq 0$, and $b \geq 0$, such that $ab < 1$, $(\varphi, f) \mapsto \varphi(\Delta_\theta)f$ is a continuous map from $\mathcal{L}_a^+ \times \mathcal{L}_b^+$ into $\mathcal{L}_c^+$, where $c = b(1-ab)^{-1}$.*

An important kind of such operators corresponds to the choice of φ being of the form $\varphi_a(z) = \exp(az)$. Every $\varphi_a(\Delta_\theta)$, $a \in \mathbb{C}$ maps $\mathcal{A}_0$ into itself (see Corollary 1.6). Moreover, the family $\{\varphi_a(\Delta_\theta)\ , a \in \mathbb{C}\}$, defined on $\mathcal{A}_0$, has a group property

$$\varphi_a(\Delta_\theta)\varphi_{a'}(\Delta_\theta) = \varphi_{a+a'}(\Delta_\theta), \tag{1.18}$$

that may be proved on the base of (1.14). For $a \geq 0$, the function φ_a belongs to $\mathcal{L}^+$. In this case the operator $\varphi_a(\Delta_\theta)$ has the following integral representation.

Proposition 1.5 *For every $\theta \geq 0$, $a > 0$, $b \geq 0$, such that $ab < 1$, and for all $f \in \mathcal{A}_b$,*

$$\begin{aligned} (\exp(a\Delta_\theta)f)(z) &= \exp(-\frac{z}{a}) \int_0^\infty s^{\theta-1} e^{-s} w_\theta \left(\frac{sz}{a}\right) f(as) ds \qquad (1.19) \\ &= \int_0^\infty K_\theta \left(\frac{z}{a}s\right) f(as) s^{\theta-1} e^{-s} ds, \end{aligned}$$

with

$$K_\theta(z,s) \stackrel{\rm def}{=} e^{-z} w_\theta(zs), \quad w_\theta(z) \stackrel{\rm def}{=} \sum_{k=0}^{\infty} \frac{z^k}{\gamma_\theta(k)}. \tag{1.20}$$

Remark 1.1 *The integral kernel just appeared has the following expansion in terms of the Laguerre polynomials (1.12), (1.13)*

$$K_\theta(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \frac{\tilde{L}_n^{(\theta-1)}(s)}{\Gamma(\theta+n)}.$$

Therefore, $\{\tilde{L}_n^{(\theta-1)}(s)/\Gamma(\theta+n)\}$ are the generalized Appell polynomials with respect to the kernel K_θ (see [3], p. 17).

Proof of Proposition 1.5. By Corollary 1.6, $\exp(a\Delta_\theta)$ is a continuous operator on $\mathcal{A}_b$, thus the left hand side of (1.19) is well defined giving a function from $\mathcal{A}_c$, $c = b(1-ab)^{-1}$. Then statement (i) of Theorem 1.1 and the continuity of the operator imply that the representation (1.19) needs to be proved only for $f(z) = z^m$. The definition (1.14) yields

$$\exp(a\Delta_\theta) z^m = \sum_{n=0}^{m} \frac{a^n}{n!} \frac{\gamma_\theta(m)}{\gamma_\theta(m-n)} z^{m-n}. \tag{1.21}$$

It is not difficult to show that, for this choice of f, the summation and the integration in the right hand side of (1.19) may be interchanged, which gives

$$\mathrm{RHS}(1.19) = \exp(-\frac{z}{a}) \sum_{k=0}^{\infty} \frac{a^{m-k} z^k}{\gamma_\theta(k)} \Gamma(\theta+m+k)$$

By means of the following Vandermonde–like convolution identity (the proof see below)

$$\frac{\Gamma(z+m+k)}{\Gamma(z+m)\Gamma(z+k)} = \sum_{n=0}^{\min(m,k)} \binom{m}{n}\binom{k}{n} \frac{n!}{\Gamma(z+n)}; \ z \in \mathbb{C}; \ m,k, \in \mathbb{N}, \tag{1.22}$$

we have:

$$\mathrm{RHS}(1.19) \quad = \quad \exp(-\frac{z}{a}) \sum_{k=0}^{\infty} \frac{a^{m-k} z^k}{k!} \Gamma(\theta+m) \frac{\Gamma(\theta+m+k)}{\Gamma(\theta+m)\Gamma(\theta+k)}$$

$$
\begin{aligned}
&= \exp(-\frac{z}{a})\sum_{k=0}^{\infty}\frac{a^{m-k}z^k}{k!}\Gamma(\theta+m) \\
&\quad \sum_{n=0}^{\min(m,k)}\frac{m!}{n!(m-n)!}\frac{k!}{n!(k-n)!}\frac{n!}{\Gamma(\theta+n)} \\
&= \exp(-\frac{z}{a})\sum_{n=0}^{m}\frac{a^{m-n}z^n m!}{n!(m-n)!}\frac{\Gamma(\theta+m)}{\Gamma(\theta+n)}\sum_{k=n}^{\infty}\frac{a^{n-k}z^{k-n}}{(k-n)!} \\
&= \sum_{n=0}^{m}\frac{a^{m-n}z^n m!}{(m-n)!n!}\frac{\Gamma(\theta+m)}{\Gamma(\theta+n)} = \mathrm{RHS}(1.21) = \mathrm{LHS}(1.19).
\end{aligned}
$$

■

The assertion just proved may be used to extend the operator $\exp(a\Delta_\theta)$. In this case one ought to consider the representation (1.19) as a definition of the extended operator. Here its following property – a kind of the operation rule (c.f. [4]) – may be useful.

Proposition 1.6 *Given $a > 0$ and $u \in \mathbb{R}$, let b satisfy $0 \le b < -u + 1/a$. Then, for every $g \in \mathcal{A}_b$, the operator (1.19) may be applied to the function*

$$f(z) = \exp(uz)g(z), \tag{1.23}$$

yielding

$$(\exp(a\Delta_\theta)f)(z) \;=\; (1-ua)^{-\theta}\exp\left(\frac{uz}{1-ua}\right)h(z), \tag{1.24}$$

where

$$
\begin{aligned}
h(z) &= \left[\exp\left(\frac{a}{1-ua}\Delta_\theta\right)g\right]\left(\frac{z}{(1-ua)^2}\right) \qquad (1.25)\\
&= \exp[a(1-ua)\Delta_\theta]\left[g\left(\frac{z}{(1-ua)^2}\right)\right].
\end{aligned}
$$

Moreover, $h \in \mathcal{A}_c$ with $c = b(1-ua)^{-1}[1-a(u+b)]^{-1}$.

Remark 1.2 *For a negative u, the above statement extends the considered operator on $\mathcal{A}_d$ with $d < |u| + 1/a$, but this obviously does not exhaust all possible extensions. The right hand side of (1.19) may be used to define*

an integral operator in the Hilbert space $L^2(I\!\!R_+, \mu_\theta)$ *possessing the kernel* $K_\theta(z,s)$ *(1.20). Here* $I\!\!R_+ = [0, +\infty)$ *and* μ_θ*,* $\theta > 0$ *is the Euler measure*

$$\mu_\theta(ds) \stackrel{\text{def}}{=} \frac{1}{\Gamma(\theta)} s^{\theta-1} e^{-s} ds.$$

We construct such and other similar extensions in a separate work.

Proof of Proposition 1.6. For f given by (1.23), one has

$$\begin{aligned}(\exp(a\Delta_\theta)f)(z) & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad(1.26)\\ = \quad & \exp(-\frac{z}{a}) \int_0^{+\infty} s^{\theta-1} \exp[-s(1-ua)] w_\theta \left(\frac{sz}{a}\right) g(as) ds.\end{aligned}$$

A simple calculation yields

$$\begin{aligned}\text{RHS}(1.26) &= (1-ua)^{-\theta} \exp(-\frac{z}{a}) \\ & \int_0^{+\infty} s^{\theta-1} e^{-s} w_\theta \left(\frac{s(1-ua)}{a} \frac{z}{(1-ua)^2} \right) g(\frac{as}{1-ua}) ds, \\ &= (1-ua)^{-\theta} \exp(-\frac{z}{a}) \exp\left(\frac{1-ua}{a} \frac{z}{(1-ua)^2} \right) \\ & \left[\exp\left(\frac{a}{1-ua} \Delta_\theta \right) g \right] \left(\frac{z}{(1-ua)^2} \right),\end{aligned}$$

which gives (1.24) and the first part of (1.25). The second part of the latter may be obtained by a change of variables. The final part of the statement follows directly from Corollary 1.6. ■

Employing the extension by (1.19) we obtain an extended form of Theorem 1.3.

Theorem 1.4 *For every* $a > 0$*,* $\theta \geq 0$*, the operator* $\exp(a\Delta_\theta)$ *(1.19) maps:*

(i) $\mathcal{L}_b^+$ *into* $\mathcal{L}_c^+$*, with* $0 \leq b < 1/a$ *and* $c = b(1-ab)^{-1}$*;*

(ii) $\mathcal{L}^-$ *into* $\mathcal{L}^-$*.*

Remark 1.3 *The operator (1.19) acts on the whole* $\mathcal{L}^-$ *with no growth restrictions.*

Proof of Theorem 1.4. Claim (i) is simply a repetition of Theorem 1.3, which we add here in order to describe the action of this operator on the whole $\mathcal{L}$ in one statement. To prove (ii) we observe that every $f \in \mathcal{L}^-$ may be written in the form (1.23) with $g \in \mathcal{L}_0$ and $u < 0$. Then, for every positive a, one may choose $b = 0$ and apply the operator (1.19) in accordance with Proposition 1.6. The result will be given by (1.24). Since $a(1 - ua)^{-1}$ is positive, $u(1 - ua)^{-1}$ is negative and the function

$$\left[\exp\left(\frac{a}{1 - ua}\Delta_\theta\right) g\right]\left(\frac{z}{(1 - ua)^2}\right)$$

belongs to $\mathcal{L}_0$, which means that its product with $\exp(u(1 - ua)^{-1}z)$ belongs to $\mathcal{L}^-$. ■

Now we may use the operators introduced above to solve the following initial value problem.

$$\begin{aligned} \frac{\partial f(t,z)}{\partial t} &= \theta\frac{\partial f(t,z)}{\partial z} + z\frac{\partial^2 f(t,z)}{\partial z^2}, \quad t \in I\!R_+, \ z \in \mathbb{C}, \\ f(0,z) &= g(z). \end{aligned} \tag{1.27}$$

Theorem 1.5 *For every $\theta \geq 0$ and $g \in \mathcal{E}$ having the form*

$$g(z) = \exp(-\varepsilon z)h(z), \quad h \in \mathcal{A}_0, \ \ \varepsilon \geq 0, \tag{1.28}$$

(i) *the problem (1.27) has in $\mathcal{A}_\varepsilon$ the following solution*

$$\begin{aligned} f(t,z) &= \left(\exp(t\Delta_\theta)g\right)(z) \\ &= \exp(-\frac{z}{t})\int_0^{+\infty} s^{\theta-1} w_\theta\left(\frac{zs}{t}\right) e^{-s} g(ts)ds, \quad t > 0. \end{aligned} \tag{1.29}$$

(ii) *If the initial condition g possesses (1.28) with $\varepsilon > 0$, then the solution (1.29) converges to zero when $t \to +\infty$ uniformly on compact subsets of $\mathbb{C}$.*

(iii) *If in (1.28) $h \in \mathcal{L}_0 \subset \mathcal{A}_0$, then the solution (1.29) belongs either to $\mathcal{L}_0$, for $\varepsilon = 0$, or to $\mathcal{L}^-$, for $\varepsilon > 0$.*

Claim (ii) means that the so called stabilization of the solutions holds (see e.g. [7] and [5]).

2 Proofs

2.1 Spaces of entire functions

We start with the proof of Theorem 1.1 by introducing another norms. For appropriate $f \in \mathcal{E}$ and some $b > 0$, we set

$$N_b(f) \stackrel{\text{def}}{=} \sup_{z \in \mathbb{C}} \{|f(z)| \exp(-b\,|z|)\} = \sup_{r \in \mathbb{R}_+} \{M_f(r) \exp(-br)\}, \tag{2.1}$$

where

$$M_f(r) \stackrel{\text{def}}{=} \sup_{|z| \leq r} |f(z)|\,, \quad r \in \mathbb{R}_+.$$

Obviously $N_b(.)$ is a norm on a subset of $\mathcal{E}$.

Proposition 2.1 *For given $f \in \mathcal{B}_b$, let $N_{b-\varepsilon}(f) < \infty$ with some $\varepsilon \in (0, b)$. Then there exists a constant $C(b, \varepsilon)$ such that*

$$N_b(f) \leq \|f\|_b \leq C(b, \varepsilon) N_{b-\varepsilon}(f). \tag{2.2}$$

Proof. By means of the Cauchy inequality one obtains

$$\left| f^{(k)}(0) \right| \leq \frac{k!}{r^k} M_f(r), \quad k \in \mathbb{N},\ r \in \mathbb{R}_+.$$

By the definition

$$M_f(r) \leq N_{b-\varepsilon}(f) \exp[(b-\varepsilon)r],$$

thus

$$\left| f^{(k)}(0) \right| b^{-k} \leq N_{b-\varepsilon}(f) k! \chi_k(r). \tag{2.3}$$

The function $\chi_k(r) \stackrel{\text{def}}{=} (br)^{-k} \exp[(b-\varepsilon)r]$ has the unique minimum at $r = k(b-\varepsilon)^{-1}$, hence the latter estimate would be the best possible for this value of r. We set

$$C_0(b, \varepsilon) = 1, \quad C_k(b, \varepsilon) = k! \chi_k\left(\frac{k}{b-\varepsilon}\right) = \frac{k!}{k^k}\left(1 - \frac{\varepsilon}{b}\right)^k e^k,$$

and obtain in (2.3)

$$\left| f^{(k)}(0) \right| b^{-k} \leq C_k(b, \varepsilon) N_{b-\varepsilon}(f).$$

By means of the Stirling formula, one may get convinced that the sequence $\{C_k(b,\varepsilon)\}$ is bounded. Thus we set

$$C(b,\varepsilon) \stackrel{\text{def}}{=} \sup_{k\in\mathbb{N}_0} C_k(b,\varepsilon),$$

and obtain the upper bound of $\|f\|_b$ in (2.2). To complete the proof we observe that

$$M_f(r) \leq \sum_{k=0}^{\infty} \frac{1}{k!} \left| f^{(k)}(0) \right| b^{-k} (br)^k \leq \|f\|_b \exp(br),$$

which immediately yields the lower bound in (2.2). ■

The family $\{N_b(.)\ b > a\}$ defines a topology on $\mathcal{A}_a$, which by Proposition 2.1 is equivalent to the topology $\mathcal{T}_a$ introduced above. We use this fact as follows.

Proposition 2.2 *For every $a \geq 0$, the space $\mathcal{A}_a$ possesses the properties:*

(i) *let $f \in \mathcal{A}_a$ and $g \in \mathcal{A}_b$, then their product fg belongs to $\mathcal{A}_{a+b}$;*

(ii) *let $f \in \mathcal{A}_a$ and, for some $g \in \mathcal{E}$, the function $\lambda(r, r_0) \stackrel{\text{def}}{=} \log M_g(r) - \log M_f(r + r_0)$, with some fixed $r_0 \in \mathbb{R}_+$, be bounded as a function of $r \in \mathbb{R}_+$, then g also belongs to $\mathcal{A}_a$.*

Proof. Here we define the topology $\mathcal{T}_a$ by means of the family $\{N_b(.),\ b > a\}$. The proof of (i) is obvious. The proof of (ii) is also quite simple:

$$\begin{aligned} N_b(g) &= \sup_{r\in\mathbb{R}_+} \{M_f(r + r_0)\exp[-b(r + r_0)]\exp[br_0 + \lambda(r, r_0)]\} \\ &\leq N_b(f)\exp(br_0) \sup_{r\in\mathbb{R}_+} \exp[\lambda(r, r_0)]. \end{aligned}$$

■

Proof of Theorem 1.1 and Corollaries 1.1 – 1.3. Proposition 2.2 yields that we may use here Proposition 2.7 and Proposition 2.5 of [12], which imply claim (i) and claim (ii) respectively. Corollary 1.1 follows directly from claim (ii), which also makes possible to extend Montel's property on $\mathcal{A}_a$ from $\mathcal{E}$. The proof of Corollary 1.3 may be given as follows. Both sequences are

bounded in the corresponding spaces. By means of the triangle inequality one gets

$$N_{a'+b'}(fg - f_n g_n) \leq N_{a'}(f_n) N_{b'}(g_n - g) + N_{a'}(f_n - f) N_{b'}(g),$$

which yields the converges to be proved.

■

Now let $\{f_n,\ n \in I\!N\} \subset \mathcal{L}^+$ converge in $\mathcal{E}$ to a function f, which does not vanish identically. Then by Proposition 1.2 the latter function also belongs to $\mathcal{L}^+$ and each such a function may be written in the form (1.6). Since some of the negative zeros of f_n could converge to zero, certain sequences $\{\beta_j(n),\ n \in I\!N\}$ would be unbounded and at the same time the sequence $\{C_n\}$ would converge to zero. In view of this possibility it is more convenient to rewrite (1.6) for f_n as follows

$$f_n(z) = p_n(z)\tilde{f}_n(z), \quad p_n(z) = C_n z^{l_n} \prod_{k=1}^{q_n} (z + z_k(n)), \tag{2.4}$$

$$\tilde{f}_n(z) = \exp(\alpha_n z) \prod_{j=1}^{\infty} (1 + \beta_j(n) z), \tag{2.5}$$

and suppose that the sequences $\{\beta_j(n), n \in I\!N\}$ are bounded and the sequences $\{z_k(n), n \in I\!N\}$ converge to zero. We also write

$$\begin{aligned} f(z) &= p(z)\tilde{f}(z), \quad p(z) = C z^l, &(2.6)\\ \tilde{f}(z) &= \exp(\alpha z) \prod_{j=1}^{\infty} (1 + \beta_j z). &(2.7) \end{aligned}$$

Recall that in the above representation all $\beta_j(n)$ and β_j are numbered according to the definition (1.6), i.e., $\beta_j(n) \leq \beta_{j+1}(n)$ The following statement, which describe the convergence of the sequences $\{C_n\}$, $\{\beta_j(n)\}$, $\{z_k(n)\}$, follows directly from the assumed convergence $f_n \to f$ by known Hurwitz's theorem (see [1], p. 167).

Proposition 2.3 *There exist positive integers n_*, l_*, and q_* such that $l_* + q_* = l$ and, for all $n > n_*$, $l_n = l_*$, $q_n = q_*$, all the sequences $\{z_k(n),\ n = n_* + 1, \ldots\}$, $k = 1, \ldots q_*$ converge to zero, and the sequence $\{\beta_1(n),\ n = n_* + 1, \ldots\}$ converges to β_1. If all $\beta_j = 0$, then all the sequences $\{\beta_j(n),\ n \in I\!N\}$ converge to zero.*

Proof of Theorem 1.2. By Proposition 1.2 the convergence $f_n \to f$ in $\mathcal{E}$ implies that $f \in \mathcal{L}^+$, which proves the final part of the theorem. For the functions considered, we use the forms (2.4) – (2.7). The convergences described by Proposition 2.3 implies $p_n \to p$ in $\mathcal{E}$. As a sequence of polynomials of bounded degree, the sequence $\{p_n\}$ is bounded in $\mathcal{A}_0$ hence it converges to p in $\mathcal{A}_0$. Therefore, by Corollary 1.3 it remains to prove the following convergence in $\mathcal{A}_a$

$$\tilde{f}_n \to \tilde{f}. \tag{2.8}$$

As for the parameter α in (2.6), it must be bounded $\alpha \leq a$ since $f \in \mathcal{A}_a$. We choose $\alpha = a$ – the convergence (2.8) for the other values of α will follow from the proof for this choice. The above proven convergence $p_n \to p$ and the assumed convergence in $\mathcal{E}$ of the sequence $\{f_n\}$ imply also the convergence $\tilde{f}_n \to \tilde{f}$ in $\mathcal{E}$, which yields by Weierstrass' theorem

$$\tilde{f}_n^{(k)}(0) \to \tilde{f}^{(k)}(0), \quad k \in \mathbb{N}_0, \quad n \to \infty. \tag{2.9}$$

For all $n \in \mathbb{N}$, $\tilde{f}_n(0) = \tilde{f}(0) = 1$, therefore, the functions $\psi_n(z) \stackrel{\text{def}}{=} \log \tilde{f}_n(z)$, $\psi(z) \stackrel{\text{def}}{=} \log \tilde{f}(z)$ are differentiable at $z = 0$, their derivatives of order $k \in \mathbb{N}$ may be written as polynomials of corresponding $f_n^{(l)}(0)$ or $f^{(l)}(0)$ with $l = 0, 1, \ldots, k$. Hence (2.9) yields

$$\psi_n^{(k)}(0) \to \psi^{(k)}(0), \quad k \in \mathbb{N}, \quad n \to \infty,$$

which may be written

$$\mu_k(n) \to \mu_k \stackrel{\text{def}}{=} \sum_{j=1}^{\infty} \beta_j^k, \quad k \geq 2, \tag{2.10}$$

and

$$\alpha_n + \mu_1(n) \to a + \mu_1, \tag{2.11}$$

where μ_1 is defined by (2.10) with $k = 1$ and the notation (1.8) has been used. In view of claim (ii) of Theorem 1.1 we will show the convergence (2.8) by proving the boundedness of the sequence $\{\tilde{f}_n\}$ in $\mathcal{A}_a$. To this end we use the topology on $\mathcal{A}_a$ defined by the family of norms (2.1). It is seen that, for $c > a$,

$$N_c(\tilde{f}) = \sup_{r \in \mathbb{R}_+} \{\exp[-(c-a)r] \prod_{j=1}^{\infty} (1 + \beta_j r)\}. \tag{2.12}$$

Since all $f_n \in \mathcal{A}_a$, the sequence of nonnegative parameters $\{\alpha_n\}$ is bounded $\alpha_n \leq a$. Similarly to (2.12), we obtain

$$N_c(\tilde{f}_n) = \sup_{r \in \mathbb{R}_+} \{\exp[-(c-\alpha_n)r] \prod_{j=1}^{\infty} (1 + \beta_j(n)r)\}, \quad c > \alpha_n. \tag{2.13}$$

First we consider the simplest situations. Suppose that all $\beta_j(n)$ and β_j equal to zero. Then (2.11) implies $\alpha_n \to \alpha$ and (2.8) obviously holds. Suppose now that all $\beta_j = 0$ and all $\tilde{f}_n$, except maybe a finite number of such functions, have finitely many $\beta_j(n)$ different from zero. Then by means of Proposition 2.3 one can easily show that

$$\prod_{j=1}^{m_n} (1 + \beta_j(n)z) \to 1, \quad n \to \infty$$

in $\mathcal{A}_0$, as it took place with the convergence $p_n \to p$. Then again $\exp(\alpha_n z) \to \exp(az)$ in $\mathcal{A}_a$. The remaining situations are more complicated.

The boundedness of the sequence $\{\tilde{f}_n\}$ in $\mathcal{A}_a$ is proven by showing that the sequence of norms $\{\mathbb{N}_c(\tilde{f}_n)\}$ is bounded for all $c > a$. Let us start with the evaluation of $N_c(\tilde{f})$. If all $\beta_j = 0$, then $N_c(\tilde{f}) = 1$ for all $c > a$. In the case of nonzero β_j, we find the point $r_c \in \mathbb{R}_+$ where the supremum in (2.12) is achieved. It may be done by solving the equation

$$c = a + \sum_{j=1}^{\infty} \frac{\beta_j}{1 + \beta_j r} \stackrel{\text{def}}{=} \varphi(r). \tag{2.14}$$

Except for the case where all $\beta_j = 0$, which we consider at the end of this proof, φ is a monotone decreasing function on $\mathbb{R}_+$. Since the series $\sum \beta_j$ converges (see (1.6)), the second term in (2.14) tends to zero when $r \to +\infty$, hence φ takes on $\mathbb{R}_+$ all values from $(a, a + \mu_1]$. Thus, for $c > a + \mu_1$, $N_c(\tilde{f}) = 1$. For $c \in (a, a + \mu_1]$,

$$N_c(\tilde{f}) = \exp[-(c-a)r_c] \prod_{j=1}^{\infty} (1 + \beta_j r_c), \tag{2.15}$$

where r_c is the unique solution of the equation (2.14). Similarly one obtains from (2.13) the following equation

$$c = \alpha_n + \sum_{j=1}^{\infty} \frac{\beta_j(n)}{1 + \beta_j(n)r} \stackrel{\text{def}}{=} \varphi_n(r), \tag{2.16}$$

where φ_n is also a monotone decreasing function taking on $I\!R_+$ all values from the interval $(\alpha_n, \alpha_n + \mu_1(n)]$. Thus, for $c > \alpha_n + \mu_1(n)$, $N_c(\tilde{f}_n) = 1$. For $c \in (\alpha_n, \alpha_n + \mu_1(n)]$, the equation (2.16) has the unique solution r_n, which defines the norm

$$N_c(\tilde{f}_n) = \exp[-(c - \alpha_n) r_n] \prod_{j=1}^{\infty} (1 + \beta_j(n) r_n). \tag{2.17}$$

Obviously each φ_n may be analytically continued on the complex half-plane $A_\varepsilon \stackrel{\text{def}}{=} \mathrm{Re} r \geq -\varepsilon$ with some $\varepsilon > 0$, which obeys the conditions

$$\varepsilon \sup_{n > n_*} \beta_1(n) \stackrel{\text{def}}{=} \varepsilon\beta < 1. \tag{2.18}$$

Such supremum exists by Proposition 2.3. Let us show that the sequence $\{\varphi_n, \ n = n_* + 1, \ldots\}$ is bounded on A_ε. Set $r + \varepsilon = z = x + iy$, then

$$\varphi_n(r) = \alpha_n + \sum_{j=1}^{\infty} \frac{\beta_j^*(n)}{1 + \beta_j^*(n) z}, \quad \beta_j^*(n) \stackrel{\text{def}}{=} \frac{\beta_j(n)}{1 - \varepsilon \beta_j(n)}. \tag{2.19}$$

Further, for $r \in A_\varepsilon$, $x \geq 0$ and $y \in I\!R$ and one readily obtains

$$\begin{aligned} 0 \ &< \ \mathrm{Re}\varphi_n(r) = \alpha_n + \sum_{j=1}^{\infty} \frac{\beta_j^*(n)(1 + \beta_j^*(n)x)}{(1 + \beta_j^*(n)x)^2 + (\beta_j^*(n)y)^2} \\ &\leq \ \alpha_n + \sum_{j=1}^{\infty} \frac{\beta_j^*(n)}{1 + \beta_j^*(n)x} \leq \alpha_n + \sum_{j=1}^{\infty} \beta_j^*(n) \\ &\leq \ a + \frac{\mu_1(n)}{1 - \varepsilon\beta} \leq a + \frac{1}{1 - \varepsilon\beta} \sup_{n > n_*} \mu_1(n). \end{aligned}$$

In view of (2.11) the latter supremum exists. Similarly

$$\begin{aligned} |\mathrm{Im}\varphi_n(r)| \ &= \ \sum_{j=1}^{\infty} \frac{[\beta_j^*(n)]^2 \, |y|}{(1 + \beta_j^*(n)x)^2 + (\beta_j^*(n)y)^2} \leq \sum_{j=1}^{\infty} \frac{\beta_j^*(n) \, |y|}{1 + (\beta_j^*(n)y)^2} \beta_j^*(n) \\ &\leq \ \sum_{j=1}^{\infty} \beta_j^*(n) \leq \frac{1}{1 - \varepsilon\beta} \sup_{n > n_*} \mu_1(n). \end{aligned}$$

Therefore, by Montel's compactness criterium ([1], p.120) the sequence $\{\varphi_n\}$ is relatively compact on A_ε. It is easily seen that the convergence established

by (2.9), (2.10) implies $\varphi_n^{(k)}(0) \to \varphi^{(k)}(0)$ for all $k \in I\!N_0$. This is enough for the convergence $\varphi_n \to \varphi$ and also for the derivatives $\varphi_n' \to \varphi'$, uniformly on compact subsets of A_ε (see [1], p.121). But φ' is strictly negative for all $r \in I\!R_+$, which means that, for any $R > r_c$, there exists $C(R)$ such that

$$0 < C(R) \leq \inf_{r\in[0,R]} |\varphi_n'(r)|\,,$$

for all sufficiently large n. Here we assume that n_* is such that this estimate holds for all $n > n_*$. A simple use of the proven uniform convergences yields that $r_n \to r_c$ with the estimate

$$|r_n - r_c| \leq \frac{1}{C(R)} \sup_{r\in[0,R]} |\varphi_n(r) - \varphi(r)|\,.$$

This yields the boundedness of the sequence of norms $N_c(\tilde{f}_n)$, that was to be proven. It remains to consider the case where $\tilde{f}(r) = \exp(ar)$. Here $\varphi(r) \equiv a$ and by Proposition 2.3 all $\beta_j(n)$ tend to zero. Moreover, (2.10) and (2.11) yield in this case $\alpha_n + \mu_1(n) \to a$ and $\mu_k(n) \to 0$, $k \geq 2$. Then we write

$$\varphi_n(r) = \alpha_n + \mu_1(n) + \omega_n(r),$$

with

$$|\omega_n(r)| \leq \max\{r\mu_2(n), r^2\mu_3(n)\},$$

which yields $\varphi_n(r) \to a$ uniformly on every $[0, R]$. This means $N_c(\tilde{f}_n) \to 1$. ∎

Proof of Corollaries 1.4 , 1.5. A compact subset of $\mathcal{E}$ is bounded and closed in $\mathcal{E}$. Then, being a subset of $\mathcal{L}_a^+$, it is bounded and closed in $\mathcal{A}_a$ provided it does not contain $f \equiv 0$ by Theorem 1.2. Hence it is compact in $\mathcal{A}_a$ in view of its Montel's property. By Proposition 1.2, every sequence of polynomials from $\mathcal{P}^+$ converges in $\mathcal{E}$ to some $f \in \mathcal{L}_a^+$ $a \geq 0$, and every such a function is a limit in $\mathcal{E}$ of a sequence of polynomials from $\mathcal{P}^+$. The sequence of polynomials obviously is a subset of $\mathcal{A}_a$ with any $a \geq 0$. By the above theorem this sequence converges to f in $\mathcal{A}_a$. ∎

2.2 Operators

The proof of Theorem 1.3 is divided on several steps. Below we will need a tool to control the distribution of zeros of certain holomorphic functions.

Thus we begin with its construction. Introduce

$$A = \{z \in \mathbb{C} \mid \mathrm{Re} z > 0\}, \quad \bar{A} = \{z \in \mathbb{C} \mid \mathrm{Re} z \geq 0\}. \tag{2.20}$$

Lemma 2.1 *Let Q_0 and Q_1 be respectively a holomorphic function in $\bar{A}$ and a polynomial in a single complex variable. If*

$$R(v, w) \stackrel{\mathrm{def}}{=} Q_0(w) + vQ_1(w) \neq 0, \tag{2.21}$$

whenever v, $w \in \bar{A}$, then

$$S(z) \stackrel{\mathrm{def}}{=} Q_0(z) + Q_1'(z) \neq 0, \tag{2.22}$$

whenever $z \in \bar{A}$.

Proof. It is no need to prove the statement in the trivial case $Q_1 \equiv 0$. In the nontrivial case the assumed property of the function R yields that $Q_0(w) = R(0, w)$ does not vanish whenever $w \in \bar{A}$. For $v \in \bar{A} \setminus \{0\}$, we rewrite (2.21) as follows

$$R(v, w) = vT(v^{-1}, w), \quad T(\varepsilon, w) = \varepsilon Q_0(w) + Q_1(w). \tag{2.23}$$

Then $T(\varepsilon, w) \neq 0$, for $\varepsilon \in \bar{A} \backslash \{0\}$ and $w \in \bar{A}$. The above facts together with Rouché's theorem ([1], p. 167) imply that $Q_1(w) \neq 0$ for $w \in A$. Let us decompose $\bar{A}$ onto the following subsets

$$B_0 = \{z \in \bar{A} \mid Q_1(z) = 0\}, \quad B_1 = \bar{A} \setminus B_0. \tag{2.24}$$

Recall that Q_1 is a polynomial, which does not vanish on A, thus B_0 is a part (maybe empty) of the finite set of Q_1 zeros, that is

$$B_0 = \{z_1, \ldots, z_m\}, \quad \mathrm{Re} z_j = 0, \; j = 1, \ldots, m. \tag{2.25}$$

For $z \in \bar{A}$, we set

$$S(z) = Q_s(z)T_s(z), \qquad z \in B_s; \; s = 0, \; 1; \tag{2.26}$$

with

$$T_0(z) = 1 + \frac{Q_1'(z)}{Q_0(z)}, \quad T_1(z) = \frac{Q_0(z)}{Q_1(z)} + \frac{Q_1'(z)}{Q_1(z)}.$$

Let us prove now that $T_s(z) \neq 0$, when $z \in B_s$.

(a) For every $z \in B_0$, there exist two possibilities:

$$\text{(i)}\ \ \mathrm{Im}\frac{Q_1'(z)}{Q_0(z)} \neq 0; \qquad \text{(ii)}\ \ \mathrm{Im}\frac{Q_1'(z)}{Q_0(z)} = 0. \tag{2.27}$$

The first one immediately yields $T_0(z) \neq 0$. In the second case we show that the real part of T_0 is not less than 1. To this end let us consider the following equation

$$T(\varepsilon, w) = 0; \quad \varepsilon \in \mathbb{C}, \quad w \in \bar{A}, \tag{2.28}$$

or

$$\varepsilon Q_0(w) = -Q_1(w); \quad \varepsilon \in \mathbb{C}, \quad w \in \bar{A}.$$

Recall that Q_0 is a holomorphic function on $\bar{A}$, which does not vanish there. Hence, for $w \in \bar{A}$, the latter equation has the following solution

$$\varepsilon = \varepsilon(w) \stackrel{\text{def}}{=} -\frac{Q_1(w)}{Q_0(w)},$$

which is a holomorphic function at any point of $\bar{A}$, possessing isolated zeros in B_0. Therefore, for every $z \in B_0$, there exists a neighborhood of this point where one may write

$$\varepsilon(w) \;\;=\;\; -\frac{Q_1'(z)}{Q_0(z)}(w - z) + o(|w - z|).$$

Having in mind (see (2.25)) that, for $z \in B_0$, $\mathrm{Re}z = 0$ and that the second possibility in (2.27) is considered, one gets

$$\mathrm{Re}\varepsilon(w) \;\;=\;\; -\mathrm{Re}\frac{Q_1'(z)}{Q_0(z)}\mathrm{Re}w + o(|w - z|). \tag{2.29}$$

From the assumption of this lemma and from the definition (2.23) we know that $T(\varepsilon, w)$ does not vanish whenever $\varepsilon \in \bar{A} \setminus \{0\}$ and $w \in \bar{A}$. On the other hand, $\varepsilon(w)$ is a solution of the equation (2.28) hence the values of the function $\varepsilon(w)$ on B_1 should have negative real parts only, i.e.

$$\mathrm{Re}\varepsilon(w) < 0, \quad \text{for} \ \ w \in B_1.$$

The latter yields in turn in (2.29)

$$0 \geq \lim_{w \to z} \frac{\mathrm{Re}\varepsilon(w)}{\mathrm{Re}w} = -\mathrm{Re}\frac{Q_1'(z)}{Q_0(z)}. \tag{2.30}$$

Thus

$$\mathrm{Re}T_0(z) = T_0(z) = 1 + \mathrm{Re}\frac{Q_1'(z)}{Q_0(z)} \geq 1,$$

that was to be shown in the case (ii) in (2.27). In what follows, in both cases $T_0(z) \neq 0$ whenever $z \in B_0$.

(b) Now we prove that T_1 does not vanish on B_1. To this end we write $T_1(z) = \vartheta(z) + t(z)$, with

$$\vartheta(z) = \frac{Q_0(z)}{Q_1(z)}; \quad t(z) = \frac{Q_1'(z)}{Q_1(z)} = \sum_{j=1}^{M} \frac{1}{z - z_j} = \sum_{j=1}^{M} \frac{\bar{z} - \bar{z}_j}{|z - z_j|^2},$$

where z_j, $j = 1, \ldots, M$ belong either to B_0 (see (2.25)) or to $\mathbb{C} \setminus \bar{A}$. In any case $\mathrm{Re}z_j \leq 0$, thus, for $z \in B_1$, $\mathrm{Re}t(z) \geq 0$. Hence it suffices to prove that $\mathrm{Re}\vartheta(z) > 0$ whenever $z \in B_1$. Rewrite (2.21) in the form

$$R(v, w) = Q_1(w)[\vartheta(w) + v]; \quad v \in \bar{A}, \quad w \in B_1. \tag{2.31}$$

Suppose $\mathrm{Re}\vartheta(w) \leq 0$, for some $w \in B_1$. Then one may set in (2.31) $v = -\mathrm{Re}\vartheta(w) - i\mathrm{Im}\vartheta(w)$ and obtain that $R(v, w) = 0$, for $v \in \bar{A}$ and $w \in B_1 \subset \bar{A}$, which is contradictory to the assumption (2.21). That means $T_1(z) \neq 0$. ■

Remark 2.1 *The above lemma is a generalization of the similar statement proved by E. H. Lieb and A. D. Sokal in [9], where the case with both Q_0 and Q_1 being polynomials was considered. We use the possibility to take Q_0 being a meromorphic function below.*

First we prove a simple corollary of Lemma 2.1

Lemma 2.2 *Let Q_0 and Q_1 be respectively a holomorphic function in A and a polynomial in a single complex variable. If*

$$R(v, w) \stackrel{\mathrm{def}}{=} Q_0(w) + vQ_1(w) \neq 0, \tag{2.32}$$

whenever v, $w \in A$, then

$$S(z) \stackrel{\mathrm{def}}{=} Q_0(z) + Q_1'(z) \neq 0, \tag{2.33}$$

whenever $z \in A$, or else $S(z) \equiv 0$.

Proof For arbitrary $\delta > 0$, we set

$$R_\delta(v,w) \stackrel{\text{def}}{=} R(v+\delta, w+\delta); \quad S_\delta(z) \stackrel{\text{def}}{=} S(z+\delta).$$

Clearly $R_\delta(v,w) \neq 0$ whenever $v,\ w \in \bar{A}$, then by Lemma 2.1 one gets $S_\delta(z) \neq 0$, whenever $z \in \bar{A}$. For $\delta \searrow 0$, $S_\delta(z)$ is uniformly convergent on compact subsets of A to $S(z)$. Thus Hurwitz's theorem yields $S(z) \neq 0$ on A, or else $S(z) \equiv 0$. ∎

Now we use the mentioned possibility to take Q_0 being a meromorphic function on $\bar{A}$.

Lemma 2.3 *Let P, Q, and Q_1 be polynomials in a single complex variable. Suppose that P does not vanish on A and*

$$Q(w) + P(w)vQ_1(w) \neq 0, \tag{2.34}$$

whenever v, $w \in A$. Then either

$$S(z) \stackrel{\text{def}}{=} Q(z) + P(z)Q_1'(z) \neq 0, \tag{2.35}$$

whenever $z \in A$, or else $S(z) \equiv 0$.

Proof Since P does not vanish on A, we get from (2.34)

$$\frac{Q(w)}{P(w)} + vQ_1(w) \neq 0,$$

for v, w belonging to A. Setting $Q_0(w) = Q(w)/P(w)$, we get from Lemma 2.2 that either

$$\hat{S}(z) \stackrel{\text{def}}{=} Q_0(z) + Q_1'(z) \neq 0,$$

whenever $z \in A$, or else $\hat{S}(z) \equiv 0$. The former implies (2.35). ∎

Now we are at a position to study our map Δ_θ

Lemma 2.4 *For arbitrary nonnegative κ and θ, $\kappa + \Delta_\theta$ maps $\mathcal{L}^+$ into $\mathcal{L}^+$.*

Proof. For the continuous operator $\kappa + \Delta_\theta : \mathcal{E} \to \mathcal{E}$, it suffices to prove the stated property on a $\mathcal{T}_C$–dense subset of $\mathcal{L}^+$. A proper choice of such subset is $\mathcal{P}^+$. Then the statement of the lemma is equivalent to the claim that the polynomial $(\kappa + \Lambda_\theta)q(z)$ with

$$\Lambda_\theta = \left(\theta + \frac{z}{2}D\right)\left(\frac{1}{2z}D\right), \quad q(z) = p(z^2), \tag{2.36}$$

does not vanish on the set A introduced in (2.20). Since p is a polynomial with real nonpositive zeros only, the polynomial q can be written as follows

$$q(z) = q_0 \prod_{j=1}^{m} (q_j + z^2), \quad m = \deg p, \quad q_j \geq 0. \tag{2.37}$$

It is no need to consider the trivial case of constant p. Consider the simplest nontrivial case where all $q_j = 0, \quad j = 1, \ldots, m$ and $q_0 \neq 0$, that is

$$q(z) = q_0 z^{2m}, \quad m \geq 1.$$

Then one gets

$$(\kappa + \Lambda_\theta) q(z) = [\kappa z^2 + m(\theta + m - 1)] q_0 z^{2(m-1)},$$

that obviously does not vanish on A. From now on we suppose that in the product in (2.37) there is at least one positive q_j. Then one may write

$$Dq(z) = 2zq(z)r(z),$$

where

$$r(z) = \sum_{j=1}^{m} \frac{1}{q_j + z^2}.$$

Then

$$\begin{aligned} (\kappa + \Lambda_\theta) q(z) &= \left[\kappa + \left(\theta + \frac{z}{2} D\right)\left(\frac{1}{2z} D\right)\right] q(z) \\ &= \kappa q(z) + \theta q(z) r(z) + \frac{z}{2} D[q(z) r(z)]. \end{aligned}$$

We set

$$Q(z) \stackrel{\text{def}}{=} q(z)(\kappa + \theta r(z)), \quad P(z) \stackrel{\text{def}}{=} \frac{z}{2}, \quad Q_1(z) \stackrel{\text{def}}{=} q(z) r(z). \tag{2.38}$$

Let us show that

$$R(v, w) \stackrel{\text{def}}{=} Q(w) + P(w) v Q_1(w) \neq 0, \tag{2.39}$$

whenever v, $w \in A$. To this end we rewrite the latter

$$R(v, w) = \frac{1}{2}(2\theta + vw) q(w) \left[\frac{2\kappa}{2\theta + vw} + r(w)\right].$$

Obviously $(2\theta + vw) \neq 0$ whenever v, $w \in A$. The same property possesses also $q(w) = p(w^2)$, $p \in \mathcal{P}^+$. Therefore, the eventual vanishing of $R(v, w)$ would imply

$$\frac{2\kappa}{2\theta + vw} + \sum_{j=1}^{m} \frac{1}{q_j + w^2} = 0$$

or equivalently

$$\frac{2\kappa(2\theta + \bar{v}\bar{w})}{|2\theta + vw|^2} + \sum_{j=1}^{m} \frac{q_j + \bar{w}^2}{|q_j + w^2|^2} = 0. \tag{2.40}$$

Introduce

$$A(v, w) \stackrel{\text{def}}{=} \frac{4\kappa\theta}{|2\theta + vw|^2} + \sum_{j=1}^{m} \frac{q_j}{|q_j + w^2|^2} > 0$$

$$B(v, w) \stackrel{\text{def}}{=} \frac{2\kappa}{|2\theta + vw|^2} \geq 0, \quad C(v, w) \stackrel{\text{def}}{=} \sum_{j=1}^{m} \frac{1}{|q_j + w^2|^2} > 0$$

and rewrite (2.40) as follows

$$C\bar{w}^2 + B\bar{v}\bar{w} + A = 0.$$

Thus

$$\bar{v} = -\frac{A}{B\,|w|^2} w - \frac{C}{B}\bar{w}, \quad B > 0,$$

or

$$\bar{w}^2 = -\frac{A}{C} < 0, \quad B = 0.$$

The above expressions imply the following conclusions

$$\mathrm{Re}v = -\left(\frac{A}{B\,|w|^2} + \frac{C}{B}\right)\mathrm{Re}w < 0,$$

or

$$\mathrm{Re}w = 0.$$

Both ones run in counter with the assumption that v and w belong to A. This means $R(v, w) \neq 0$, whenever v, $w \in A$. By means of Lemma 2.3 one concludes that $(\kappa + \Lambda_\theta)q(z) \neq 0$, for $z \in A$. ∎

Proof of Theorem 1.3. It suffices to show that for any $\varphi \in \mathcal{L}_a^+$ and $f \in \mathcal{L}_b^+$, the function $\varphi(\Delta_\theta)f$ belongs to $\mathcal{L}^+$. To this end we choose a sequence

$\{\varphi_n,\ n \in I\!N\} \subset \mathcal{P}^+$, converging to φ in $\mathcal{A}_a$ (see Corollary 1.5). The fact $\varphi_n \in \mathcal{P}^+$ implies

$$\varphi_n(z) = \phi_n \prod_{j=1}^{m_n} (\kappa_{j,n} + z), \quad \phi_n \neq 0, \quad \kappa_{j,n} \geq 0.$$

Then

$$g_n(z) \stackrel{\text{def}}{=} \varphi_n(\Delta_\theta) f(z) = \phi_n \prod_{j=1}^{m_n} (\kappa_{j,n} + \Delta_\theta) f(z).$$

Thus Lemma 2.4 yields $g_n(z) \in \mathcal{L}^+$. Corollary 1.6 implies that $\{g_n\}$ converges to $\varphi(\Delta_\theta) f$ in $\mathcal{A}_c$. The latter yields $\varphi(\Delta_\theta) f \in \mathcal{L}^+$. ■

Proof of Theorem 1.5. Consider the operator valued function $(0, t_0) \ni t \mapsto \varphi_t(\Delta_\theta) \in \mathbf{B}(\mathcal{B}_b, \mathcal{B}_c)$, where the latter is the Banach space of all linear bounded operators between the Banach spaces $\mathcal{B}_b$, $\mathcal{B}_c$ with $c = b(1 - t_0 b)$, $t_0 b < 1$. One may easily show that this function is continuous and differentiable in the norm-topology and its derivative is

$$\varphi_t'(\Delta_\theta) = \Delta_\theta \varphi_t(\Delta_\theta).$$

Therefore, for $t \in (0, 1/\varepsilon)$, one has

$$\frac{\partial f(t,z)}{\partial t} = \Delta_\theta \left(\exp(t\Delta_\theta) g\right)(z),$$

which proves the first line in (1.29). The second line, which gives the extension of the solution to all positive values of t, is easily obtained from the first one by means of Proposition 1.5. Further, we substitute in (1.29) the initial condition in the form (1.28) and apply Proposition 1.6. This yields

$$\begin{aligned} f(t,z) &= (1+\varepsilon t)^{-\theta} \exp\left(-\frac{\varepsilon z}{1+\varepsilon t}\right) \left[\exp\left(\frac{t}{1+\varepsilon t}\Delta_\theta\right) h\right] \left(\frac{z}{(1+\varepsilon t)^2}\right) \\ &\stackrel{\text{def}}{=} (1+\varepsilon t)^{-\theta} \exp\left(-\frac{\varepsilon z}{1+\varepsilon t}\right) h_t \left(\frac{z}{(1+\varepsilon t)^2}\right). \end{aligned} \tag{2.41}$$

By Corollary 1.6 $h_t \in \mathcal{A}_0$, and by Theorem 1.4 $h_t \in \mathcal{L}_0$ if $h \in \mathcal{L}_0$. The former proves claim (i) and the latter does claim (iii). It remains to prove the convergence stated in (ii). The mentioned continuity of the operator $\exp(t\Delta_\theta)$ implies that in $\mathcal{A}_0$

$$\left[\exp\left(\frac{t}{1+\varepsilon t}\Delta_\theta\right) h\right] \left(\frac{z}{(1+\varepsilon t)^2}\right) \to \left\{\exp\left(\frac{1}{\varepsilon}\Delta_\theta\right) h\right\}(0). \tag{2.42}$$

Therefore, the product in (2.41) tends to zero in $\mathcal{A}_\varepsilon$ when $t \to +\infty$. ■

Proof of the identity (1.22). Assume that $k \le m$, then the left hand side of (1.22) may be brought into the following form

$$\frac{1}{\Gamma(z+k)} \prod_{l=1}^{k} (z+m+k-l),$$

which one rewrites as

$$\frac{1}{\Gamma(z+k)} \prod_{l=1}^{k} [(z+k-j_l) + (m+j_l-l)],$$

with arbitrary j_l. Then one opens the brackets [.] and transforms the product of sums into the sum of the products choosing in every term an appropriate $j_l \in \{1, \ldots, l\}$. This yields

$$\begin{aligned} & \frac{1}{\Gamma(z+k)} \sum_{n=0}^{k} \left[\prod_{l=1}^{k-n} [(z+k-l) m(m-1)\ldots(m-n+1) \binom{k}{n} \right] \\ = & \sum_{n=0}^{k} n! \binom{m}{n} \binom{k}{n} \frac{1}{\Gamma(z+k)} \prod_{l=1}^{k-n} (z+k-l) \\ = & \sum_{n=0}^{k} \binom{m}{n} \binom{k}{n} \frac{n!}{\Gamma(z+k)} = \mathrm{RHS}(1.22). \end{aligned} \tag{2.43}$$

■

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References

[1] Berenstein, C.A., Gay, R.: *Complex Variables: An Introduction.* Springer-Verlag: New York - Berlin - Heidelberg - London - Paris - Tokyo - Hong Kong - Barcelona, 1991

[2] Berenstein, C.A., Gay, R.: *Complex Analysis and Special Topics in Harmonic Analysis.* Springer-Verlag: New York - Berlin - Heidelberg - Barcelona - Budapest - Hong Kong - London - Milan - Paris - Tokyo, 1995

[3] Boas, R.P., Buck, R.C.: *Polynomial Expansions of Analytic Functions.* Springer-Verlag: Berlin - Göttingen - Heidelberg, 1964

[4] Dattoli, G., Torre, A., Carpanese, M.: *Operational Rules and Arbitrary Order Hermite Generating Function.* J. Math. Anal. Appl. **227**, (1998), 98–111

[5] Denisov, V.N.: *On Stabilization of Solutions of the Cauchy Problem for Parabolic Equations.* Nonlinear Anal.: Theory, Methods, & Appl. **30**, (1997), 123–127

[6] Iliev, L.: *Laguerre Entire Functions.* Bulgarian Academy of Sciences: Sofia, 1987

[7] Kamin (Kamenomostovskaya), S.: *On Stabilization of Solutions of the Cauchy Problem for Parabolic Equations.* Proceed. Royal Soc. Edinburgh **76A**, (1977), 43–53

[8] Levin, B.J.: *Distribution of Zeros of Entire Functions. Amer. Math. Soc.* 1964

[9] Lieb, E.H., Sokal, A.D.: *A General Lee–Yang Theorem for One–Component and Multicomponent Ferromagnets.* Commun. Math. Phys. **80**, (1981), 268–277

[10] Pólya, G., Szegö, G.: *Problems and Theorems in Analysis.I.* Springer-Verlag: Berlin - Heidelberg - New York, 1978

[11] Schaeffer, H.H.: *Topological Vector Spaces.* Springer-Verlag: New York - Heidelberg - Berlin, 1971

[12] Taylor, B.A.: *Some Locally Convex Spaces of Entire Functions.* Proc. Sympos. Pure Math. **11**, (1968), 431–466

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